Multiparameter Golay 2-complementary sequences and transforms

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Abstract. In this work, we develop a new unified approach to the so-called generalized Golay-Rudin-Shapiro (GRS) 2-complementary sequences. It based on a new generalized iteration generating construction.

Keywords: generalized complementary sequences, multiparameter Fourier-Golay-Rudin-Shapiro transforms, OFDM telecommunication systems.

1. Introduction

Binary ± 1-valued *Golay-Rudin-Shapiro* sequences (2-GRS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by Golay [1,2,3] in 1949-1951, Shapiro [4,5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of “complementary pairs" of finite sequences all of whose entries are ± 1. This was motivated by a highly non-trivial application to infrared spectrometry. Then he gave an explicit construction for binary Golay complementary pairs of length 2^m and later [3] noted that the construction implies the existence of at least $2^m m!/2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 1^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ (Turyn, [7]), but do not exist for any length N having a prime factor congruent to the modulo 4 (Eliahou et al., [8]). In 1951, H. S. Shapiro [4,5] introduced what became known, after 1963, as the “Rudin-Shapiro” polynomial pairs. Shapiro's work was entirely in pure mathematics. Budisin [9,10,11] using the work of Sivaswamy [12] gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it coincides with those given explicitly by Golay. For a survey of results on binary and nonbinary Golay complementary pairs, see Byrnes [13] and Fan, Darnel, [14], respectively. In 1999, Davis and Jedwab [15] gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed-Muller codes.

Discrete classical *Fourier-Golay-Rudin-Shapiro Transforms* (FGRST) in bases of different Golay-Rudin-Shapiro sequences can be used in many signal processing applications: multiresolution by discrete orthogonal wavelet decomposition, digital audition, digital video broadcasting, communication systems (Orthogonal Frequency Division Multiplexing - OFDM, Multi-Code-Division Multiple Access - MCDA), radar, and cryptographic systems.

For building the classical FGRSTs in bases of classical Golay-Rudin-Shapiro sequences the following actors are used: 1) the Abelian group $\mathbf{Z}_2^n$, 2) 2-point Fourier transform F_2, and 3) the complex field $\mathbf{C}$, i.e., these transforms are associated with the triple $\left(\mathbf{Z}_2^n, \mathsf{F}_2, \mathbf{C}\right)$. In this work, we develop a new approach to the so-called generalized complex-, $\mathbf{GF}(p)$-, and Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n, \mathsf{F}_2, \mathbf{C}\right)$, but with $\left(\mathbf{Z}_2^n, \left\{\mathsf{F}_2(\varepsilon_1), \mathsf{F}_2(\varepsilon_2), ..., \mathsf{F}_2(\varepsilon_n)\right\}, \mathsf{A}lg\right)$, where $\left\{\mathsf{F}_2(\varepsilon_1), \mathsf{F}_2(\varepsilon_2), ..., \mathsf{F}_2(\varepsilon_n)\right\}$ a set of arbitrary unitary (2×2)-transforms of type $\mathsf{F}_2(\varepsilon) = \begin{bmatrix} 1 & \varepsilon \\ 1 & -\varepsilon \end{bmatrix}$, (where $\varepsilon := e^{i\theta} \in \mathsf{A}lg$, $|\varepsilon| = 1$) instead of $\mathsf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathsf{A}lg$ is an algebras (for example, Clifford algebras or finite rings $\mathbf{Z}_N$, or finite Galois fields $\mathbf{GF}(q)$) instead of the complex field $\mathbf{C}$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay-Rudin-Shapiro* binary sequences) is described. In Section 3, the iteration rule for design of the Golya matrix is introduced. In Section 4, the proposed method based on new generalized iteration construction is explained.

2. The object of the study. Iteration construction of Golay matrices

We begin by describing the original Golay 2-complementary ± 1-valued sequences.

Definition 1. Let $\mathrm{com}^0(t) := \left(c_0, c_1, ..., c_{N-1}\right)$ and $\mathrm{com}^1(t) = \left(s_0, s_1, ..., s_{N-1}\right)$, where $c_i, s_i \in \mathbf{B}_2 = \{\pm 1\}$. The sequences $\mathrm{com}^0(t)$, $\mathrm{com}^1(t)$ are called the *2-complementary* ((± 1)*-valued)* or *Goley complementary pair* over $\{\pm 1\}$, if $COR^0(\tau) + COR^1(\tau) = N\delta(\tau)$, or $\left(\left|\mathrm{COM}^0(z)\right|^2 + \left|\mathrm{COM}^1(z)\right|^2\right)_{|z|=1} = N$, where $COR^0(\tau), COR^1(\tau)$ are the periodic correlation functions of $\mathrm{com}^0(t)$, $\mathrm{com}^1(t)$ and $\mathrm{COM}^0(z) = \mathsf{Z}\left\{\mathrm{com}^0(t)\right\}$, $\mathrm{COM}^1(z) = \mathsf{Z}\left\{\mathrm{com}^1(t)\right\}$ are their Z – transforms. Any sequence, which is a member of a Golay complementary pair, is called the *Golay sequence* and its Z – transform $\mathrm{COM}_k(z) = \mathsf{Z}\left\{\mathrm{com}_k(t)\right\}$ is called the *Golay-Shapiro-Rudin polynomial (GSRP).*

We use two symbols $\boldsymbol{\alpha}_n \in [0, 2^{n-1}-1] = \mathbf{Z}_{2^n}$ and $\mathbf{t}_n \in [0, 2^{n-1}-1] = \mathbf{Z}_{2^n}$ for numeration of Golay sequences and discrete time, respectively. For integer $\boldsymbol{\alpha}_n \in [0, 2^{n-1}-1]$ and $\mathbf{t}_n \in [0, 2^{n-1}-1]$ we shall use binary codes $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$, $\vec{\mathbf{t}}_n = \left(t_1, t_2, ..., t_n\right)$, where $\alpha_i, t_i \in \{0,1\}$, $i = 1, 2, ..., n$.

Let $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$ and $\vec{\mathbf{t}}_n = \left(t_1, t_2, ..., t_n\right)$ be binary codes, then define

$$\boldsymbol{\alpha}_n = \left|\vec{\boldsymbol{\alpha}}_n\right| = \left|\left(\alpha_1, \alpha_2, ..., \alpha_n\right)\right| = \sum_{i=1}^{n} \alpha_{n-i+1} 2^{i-1},\ \mathbf{t}_n = \left|\vec{\mathbf{t}}_n\right| = \left|\left(t_1, t_2, ..., t_n\right)\right| = \sum_{i=1}^{n} t_{n-i+1} 2^{n-i}$$

be integers whose binary codes are $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$ and $\vec{\mathbf{t}}_n = \left(t_1, t_2, ..., t_n\right)$, where α_n, t_1 are less significant bits (LSB) and α_1, t_n are most significant bits (MSB) of $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$ and $\vec{\mathbf{t}}_n = \left(t_1, t_2, ..., t_n\right)$, respectively. Obviously,

$$\begin{array}{ll}
\vec{\boldsymbol{\alpha}}_1=(\alpha_1)\in\mathbf{Z}_2, & \boldsymbol{\alpha}_1=\alpha_1\in\mathbf{Z}_2,\\
\vec{\boldsymbol{\alpha}}_2=\left(\vec{\boldsymbol{\alpha}}_1,\alpha_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2=\mathbf{Z}_2^2, & \left(\boldsymbol{\alpha}_1,\alpha_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2,\\
\vec{\boldsymbol{\alpha}}_3=\left(\vec{\boldsymbol{\alpha}}_2,\alpha_3\right)\in\mathbf{Z}_2^2\times\mathbf{Z}_2=\mathbf{Z}_2^3, & \left(\boldsymbol{\alpha}_2,\alpha_3\right)\in\mathbf{Z}_{2^2}\times\mathbf{Z}_2,\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\\
\vec{\boldsymbol{\alpha}}_n=\left(\vec{\boldsymbol{\alpha}}_{n-1},\alpha_n\right)\in\mathbf{Z}_2^{n-1}\times\mathbf{Z}_2=\mathbf{Z}_2^n, & \left(\boldsymbol{\alpha}_{n-1},\alpha_n\right)\in\mathbf{Z}^1_{2^{n-1}}\times\mathbf{Z}_2\ .\\
\vec{\mathbf{t}}_1=\left(t_1\right)\in\mathbf{Z}_{2^1}, & \mathbf{t}_1=t_1\in\mathbf{Z}_2,\\
\vec{\mathbf{t}}_2=\left(\vec{\mathbf{t}}_1,t_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2=\mathbf{Z}_2^2, & \left(\mathbf{t}_1,t_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2,\\
\vec{\mathbf{t}}_3=\left(\vec{\mathbf{t}}_2,t_3\right)\in\mathbf{Z}_2^2\times\mathbf{Z}_2=\mathbf{Z}_2^3, & \left(\mathbf{t}_2,t_3\right)\in\mathbf{Z}_{2^2}\times\mathbf{Z}_2,\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\\
\vec{\mathbf{t}}_n=\left(\vec{\mathbf{t}}_{n-1},t_n\right)\in\mathbf{Z}_2^{n-1}\times\mathbf{Z}_2=\mathbf{Z}_2^n, & \left(\mathbf{t}_{n-1},t_n\right)\in\mathbf{Z}^1_{2^{n-1}}\times\mathbf{Z}_2,
\end{array}$$

where $\mathbf{Z}_2^k=\{0,1\}^k=\underbrace{\mathbf{Z}_2\times\mathbf{Z}_2\times\ldots\times\mathbf{Z}_2}_{k}$ and $\mathbf{Z}_{2^k}=\{0,1,2,\ldots,2^k-1\}$.

Let $\mathrm{com}^{[n+1]}_{(\vec{\boldsymbol{\alpha}}_n,0)}(\mathbf{t}_{n+1})$, $\mathrm{com}^{[n+1]}_{(\vec{\boldsymbol{\alpha}}_n,1)}(\mathbf{t}_{n+1})$ be a set of 2^n pairs of complementary sequences of length 2^{n+1}. Then the following matrix of depth $n+1$ has size $2^{n+1}\times 2^{n+1}$

$$\mathbf{G}^{[n+1]}_{2^{n+1}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=0}^{\mathbf{2^{n+1}-1}}\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{2^n-1}}\mathop{\boxplus}_{\alpha_{n+1}=0}^{1}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,\alpha_{n+1})}(\mathbf{t}_{n+1})=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{2^n-1}}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\mathbf{0},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{0},1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(\mathbf{1},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{1},1)}(\mathbf{t}_{n+1})\\ \hline \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ \hline \mathrm{com}^{[n+1]}_{(\mathbf{2^n-1},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{2^n-1},1)}(\mathbf{t}_{n+1})\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[n+1]}_{(0,0,\ldots,0,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(0,0,\ldots,0,1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(0,0,\ldots,1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(0,0,\ldots,1,1)}(\mathbf{t}_{n+1})\\ \hline \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ \hline \mathrm{com}^{[n+1]}_{(1,1,\ldots,1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(1,1,\ldots,1,1)}(\mathbf{t}_{n+1})\end{bmatrix}\quad(1)$$

(1)

is called the Golay matrix, where $\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}$ are a pair of complementary sequences and $\boxplus$ is the symbol of the vertical concatenation of $\left(2\times 2^{n+1}\right)$ – matrices $\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}$. For example,

$$\mathbf{G}^{[1]}_{2^1}=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\end{bmatrix}=\mathop{\boxplus}_{\boldsymbol{\alpha}_1=0}^{1}\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1),$$

$$\mathbf{G}^{[2]}_{2^2}=\left[\mathrm{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)\right]^3_{\boldsymbol{\alpha}_2,\mathbf{t}_2=\mathbf{0}}=\mathop{\boxplus}_{\boldsymbol{\alpha}_2=\mathbf{0}}^{3}\mathrm{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)=\mathop{\boxplus}_{\boldsymbol{\alpha}_1=\mathbf{0}}^{1}\begin{bmatrix}\mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[2]}_{(\mathbf{0},0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\mathbf{0},1)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(\mathbf{1},0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\mathbf{1},1)}(\mathbf{t}_2)\end{bmatrix},$$

$$\mathbf{G}_{2^3}^{[3]}=\left[\mathrm{com}_{\boldsymbol{\alpha}_3}^{[3]}(\mathbf{t}_3)\right]_{\boldsymbol{\alpha}_3,\mathbf{t}_3=\mathbf{0}}^{7}=\underset{\boldsymbol{\alpha}_3=\mathbf{0}}{\overset{7}{\boxplus}}\mathrm{com}_{\boldsymbol{\alpha}_3}^{[3]}(\mathbf{t}_3)=\underset{\boldsymbol{\alpha}_2=\mathbf{0}}{\overset{3}{\boxplus}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_2,0)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\boldsymbol{\alpha}_2,1)}^{[3]}(\mathbf{t}_3)\end{bmatrix}=\begin{bmatrix}\mathrm{com}_{(\mathbf{0},0)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{0},1)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{1},0)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{1},1)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{2},0)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{2},1)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{3},0)}^{[3]}(\mathbf{t}_3)\\ \mathrm{com}_{(\mathbf{3},1)}^{[3]}(\mathbf{t}_3)\end{bmatrix}.$$

3. Methods

The matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ is constructed by an iteration construction $\mathbf{G}_{2^1}^{[1]}\to\mathbf{G}_{2^2}^{[2]}\to....\to\mathbf{G}_{2^n}^{[n]}\to\mathbf{G}_{2^{n+1}}^{[n+1]}$. The initial matrix $\mathbf{G}_{2^1}$ is formed by starting with the Fourier-Walsh $(2\times 2)-$matrix $\mathbf{G}_{2^1}^{[1]}=\mathsf{F}_2=\begin{bmatrix}\mathrm{com}_0^{[1]}(\mathbf{t}_1)\\ \mathrm{com}_1^{[1]}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ and by repeated application of the iteration construction to pairs of rows in the matrix. Let us to suppose that we have the Golay matrix $\mathbf{G}_{2^n}^{[n]}$. We need to construct the next Golay matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ using only $\mathbf{G}_{2^n}^{[n]}$ and $\mathsf{F}_2=\mathbf{G}_{2^1}^{[1]}$. The matrix $\mathbf{G}_{2^n}^{[n]}$ have structure similar (1):

$$\mathbf{G}_{2^n}^{[n]}=\underset{\boldsymbol{\alpha}_n=0}{\overset{\mathbf{2^n-1}}{\boxplus}}\mathrm{com}_{\boldsymbol{\alpha}_n}^{[n]}(\mathbf{t}_n)=\underset{\boldsymbol{\alpha}_{n-1}=\mathbf{0}}{\overset{\mathbf{2}^{n-1}\mathbf{-1}}{\boxplus}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_{n+1})\end{bmatrix}. \quad (2)$$

For constructing $\mathbf{G}_{2^{n+1}}^{[n+1]}$ from $\mathbf{G}_{2^n}^{[n]}$ we take each complementary pair from (2) in the form of $\left[\!\left[\mathbf{com}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n)\right]\!\right]:=\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}$ and construct shifted versa of their components

$$\left[\!\left[{}^{(k)}\mathbf{Tcom}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n)\right]\!\right]=\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n+2^n(0\underset{2}{\oplus}k)\right)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n+2^n(1\underset{2}{\oplus}k)\right)\end{bmatrix}=\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}=$$

$$=\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)},\mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\right\}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix},$$

where $k=0{,}1$ and $\mathbf{T}_{\mathbf{t}_n}^{2^n s}$ is the shift operator on $2^n s$ positions in time domain: $\mathbf{T}_{\mathbf{t}_n}^{2^n s}f(\mathbf{t}_n):=f(\mathbf{t}_n+2^n\mathrm{s})$. Now we construct the general building blocks for the Golay$\left(2^{n+1}\times 2^{n+1}\right)$**-matrix** $\mathbf{G}_{2^{n+1}}^{[n]}$ **:**

$$\mathcal{F}_2\cdot\left[\!\left[{}^{(k)}\mathbf{Tcom}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n)\right]\!\right]=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)},\mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\right\}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix},$$

$$=\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\\ \mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}={}^{(k)}\mathcal{F}_2\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix},$$

where

$${}^{(k)}\mathsf{F}=\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\\ \mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\cdot\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\underset{2}{\oplus}k)},\mathbf{T}_{\mathbf{t}_n}^{2^n(1\underset{2}{\oplus}k)}\right\}.$$

Using these building blocks of $\left(2^n \times 2^n\right)$-matrix $\mathbf{G}_{2^n}^{[n]}$ we construct the Golay $\left(2^{n+1} \times 2^{n+1}\right)$-matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ according to the following iteration rule [16]:

$$\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n\right) \end{bmatrix}^{\mathsf{Z}}] \left[\begin{array}{c} {}^{(0)}\mathsf{F}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n\right) \end{bmatrix} \\ \hline {}^{(1)}\mathsf{F}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n\right) \end{bmatrix} \end{array}\right] = \left[\begin{array}{c} \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n\right) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\mathbf{I}_{\mathbf{t}_n} \end{bmatrix}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n\right) \end{bmatrix} \end{array}\right] =$$

$$= \left[\begin{array}{c} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \\ -\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \end{array}\right] = \left[\begin{array}{c} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{array}\right],$$

where

$$\begin{aligned} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) &= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) &= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) &= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) &= -\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n). \end{aligned}$$

are complementary sequences of twice length, belonging to $\mathbf{G}_{2^{n+1}}^{[n+1]}$. Hence,

$$\mathbf{G}_{2^{n+1}}^{[n+1]} = \underset{\boldsymbol{\alpha}_{n+1}=0}{\overset{2^{n+1}-1}{\boxplus}}\left[\mathrm{com}_{\boldsymbol{\alpha}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1})\right] = \underset{\boldsymbol{\alpha}_{n}=0}{\overset{2^{n}-1}{\boxplus}}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \underset{\boldsymbol{\alpha}_{n-1}=0}{\overset{2^{n}-1}{\boxplus}}\left[\begin{array}{c} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{array}\right] =$$

$$= \underset{\boldsymbol{\alpha}_{n-1}=0}{\overset{2^{n}-1}{\boxplus}}\left[\begin{array}{c} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & -\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} \end{bmatrix}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \hdashline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & -\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} \end{bmatrix}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] = \underset{\boldsymbol{\alpha}_{n-1}=0}{\overset{2^{n}-1}{\boxplus}}\left(\underset{\alpha_n=0}{\overset{1}{\boxplus}}\begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\underset{2}{\oplus}\alpha_n)} & \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\underset{2}{\oplus}\alpha_n)} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\underset{2}{\oplus}\alpha_n)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\underset{2}{\oplus}\alpha_n)} \end{bmatrix}\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}\right) =$$

$$= \underset{\boldsymbol{\alpha}_{n-1}=0}{\overset{2^{n}-1}{\boxplus}}\left(\underset{\alpha_n=0}{\overset{1}{\boxplus}}\begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\underset{2}{\oplus}\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\underset{2}{\oplus}\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\underset{2}{\oplus}\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\underset{2}{\oplus}\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}\right) =$$

$$= \underset{\boldsymbol{\alpha}_{n-1}=0}{\overset{2^{n-1}-1}{\boxplus}}\left[\begin{array}{c} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \\ -\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \end{array}\right], \tag{3}$$

or

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \sum_{\beta_n=0}^{1}(-1)^{\alpha_{n+1}\beta_n}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(\alpha_n\underset{2}{\oplus}\beta_n)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}^{[n]}(\mathbf{t}_n) =$$

$$= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n + 2^n\cdot(\alpha_n\underset{2}{\oplus}0)\right) + (-1)^{\alpha_{n+1}}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n + 2^n\cdot(\alpha_n\underset{2}{\oplus}1)\right).$$

Since $\mathbf{t}_{n+1}=\left(\mathbf{t}_n,t_{n+1}\right)$, then believing $t_{n+1}=(\alpha_n \underset{2}{\oplus} \beta_n)$, we obtain

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1})=\sum_{t_{n+1}=0}^{1}(-1)^{\left(\alpha_n \underset{2}{\oplus} t_{n+1}\right)\alpha_{n+1}}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}\left(\mathbf{t}_n+2^n\cdot t_{n+1}\right)=$$

$$=(-1)^{\alpha_n\alpha_{n+1}}\sum_{t_{n+1}=0}^{1}(-1)^{t_{n+1}\alpha_{n+1}}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}\left(\mathbf{t}_n+2^n\cdot t_{n+1}\right). \tag{4}$$

Hence,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1})=(-1)^{\left(\alpha_n \underset{2}{\oplus} t_{n+1}\right)\alpha_{n+1}}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}\left(\mathbf{t}_n\right). \tag{5}$$

It is finally recurrent relation between complementary sequences from $\mathbf{G}^{[n+1]}_{2^{n+1}}$ and $\mathbf{G}^{[n]}_{2^n}$.

Example 1.

$$\mathbf{G}^{[1]}_{2^1}\equiv \mathsf{F}_2=\begin{bmatrix}\mathrm{com}^{[1]}_{(0)}(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_{(1)}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}=\left[\mathrm{com}^{[1]}_{(\alpha_1)}(t_1)\right]=\left[(-1)^{\alpha_1 t_1}\right],$$

$$\mathbf{G}^{[2]}_{2^2}=\begin{bmatrix}\mathrm{com}^{[2]}_{(0,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(0,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,1)}(\mathbf{t}_2)\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & -1\\ 1 & 1 & -1 & 1\\ \hdashline 1 & -1 & 1 & 1\\ -1 & 1 & 1 & 1\end{bmatrix}=\left[\mathrm{com}^{[2]}_{(\alpha_1,\alpha_2)}(t_1,t_2)\right]=$$

$$=\left[\mathrm{com}^{[1]}_{(\alpha_1 \underset{2}{\oplus} t_2)}(t_1)\left(-1\right)^{(\alpha_1 \underset{2}{\oplus} t_2)\alpha_2}\right]=\left[\left(-1\right)^{t_1(\alpha_1 \underset{2}{\oplus} t_2)}\left(-1\right)^{(\alpha_1 \underset{2}{\oplus} t_2)\alpha_2}\right]=\left[\left(-1\right)^{(\alpha_0 \underset{2}{\oplus} t_1)(\alpha_1 \underset{2}{\oplus} t_2)}\left(-1\right)^{(\alpha_1 \underset{2}{\oplus} t_2)(\alpha_2 \underset{2}{\oplus} t_3)}\right]=$$

$$=\mathrm{diag}\left\{\left(-1\right)^{\alpha_1\alpha_2}\right\}\left[\left(-1\right)^{(\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2)}\right]\mathrm{diag}\left\{\left(-1\right)^{t_1 t_2}\right\},$$

where $\alpha_0,t_3\equiv 0$,

$$\mathbf{G}^{[3]}_{2^3}=\begin{bmatrix}\mathrm{com}^{[3]}_{(0,0,0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(0,0,1)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(0,1,0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(0,1,1)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(1,0,0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(1,0,1)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(1,1,0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(1,1,1)}(\mathbf{t}_3)\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & -1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1\\ \hdashline 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1\\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1\\ \hdashline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1\\ \hdashline -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1\\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1\end{bmatrix}=$$

$$=\left[\mathrm{com}^{[3]}_{(\alpha_1,\alpha_2,\alpha_3)}(t_1,t_2,t_3)\right]=\left[\mathrm{com}^{[2]}_{(\alpha_1,\alpha_2 \underset{2}{\oplus} t_3)}(t_1,t_2)\left(-1\right)^{(\alpha_2 \underset{2}{\oplus} t_3)\alpha_3}\right]=$$

$$=\left[\left(-1\right)^{(\alpha_1 \underset{2}{\oplus} t_2)(\alpha_2 \underset{2}{\oplus} t_3 \underset{2}{\oplus} t_1)}\left(-1\right)^{(\alpha_2 \underset{2}{\oplus} t_3)\alpha_3}\right]=\left[\left(-1\right)^{(\alpha_0 \underset{2}{\oplus} t_1)(\alpha_1 \underset{2}{\oplus} t_2)}\left(-1\right)^{(\alpha_1 \underset{2}{\oplus} t_2)(\alpha_2 \underset{2}{\oplus} t_3)}\left(-1\right)^{(\alpha_2 \underset{2}{\oplus} t_3)(\alpha_3 \underset{2}{\oplus} t_4)}\right]=$$

$$=\mathrm{diag}\left\{\left(-1\right)^{(\alpha_1\alpha_2 \underset{2}{\oplus} \alpha_2\alpha_3)}\right\}\cdot\left[\left(-1\right)^{\left(\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2 \underset{2}{\oplus} (\alpha_3 \underset{2}{\oplus} \alpha_1)t_3\right)}\right]\cdot\mathrm{diag}\left\{\left(-1\right)^{(t_1 t_2 \underset{2}{\oplus} t_2 t_3)}\right\},$$

where $\alpha_0,t_4\equiv 0$.

From (5) we obtain two expressions for $\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})$:

$$1)\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\left(-1\right)^{\sum_{i=1}^{n}{}^{\oplus}(\alpha_i \underset{2}{\oplus} t_{i+1})(\alpha_{i+1} \underset{2}{\oplus} t_{i+2})}, \tag{6}$$

$$2)\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\left(-1\right)^{\sum_{i=1}^{n}{}^{\oplus}\alpha_i\alpha_{i+1}}\cdot\left(-1\right)^{\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2 \underset{2}{\oplus} \sum_{i=3}^{n}{}^{\oplus}(\alpha_i \underset{2}{\oplus} \alpha_{i-2})t_i}\cdot\left(-1\right)^{\sum_{i=1}^{n}{}^{\oplus}t_i t_{i+1}}, \tag{7}$$

where $\alpha_0, t_{n+2} \equiv 0$. New sequences in (5) are orthogonal and complementary sequences.

4. Generalization

Our generalization uses the following iteration construction

$$\mathbf{G}_{2^1}^{[1]}\left[\mathsf{F}_2(\varepsilon_1)\right] \xrightarrow{\mathsf{F}_2(\varepsilon_2)} \mathbf{G}_{2^2}^{[2]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2)\right] \xrightarrow{\mathsf{F}_2(\varepsilon_3)} \mathbf{G}_{2^3}^{[3]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\mathsf{F}_2(\varepsilon_3)\right] \xrightarrow{\mathsf{F}_2(\varepsilon_4)} \ldots$$

$$\ldots \xrightarrow{\mathsf{F}_2(\varepsilon_n)} \mathbf{G}_{2^n}^{[n]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n)\right] \xrightarrow{\mathsf{F}_2(\varepsilon_{n+1})} \mathbf{G}_{2^n}^{[n+1]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n),\mathsf{F}_2(\varepsilon_{n+1})\right],$$

based on a sequence of unitary transforms:

$$\mathsf{F}_2(\varepsilon_k) = \begin{bmatrix} 1 & \varepsilon_k \\ 1 & -\varepsilon_k \end{bmatrix}, \quad \varepsilon_k = e^{i\varphi_k} \in \mathsf{A}\,lg, \quad \forall k = 1,2,\ldots,n+1.$$

For brevity let $\mathsf{U}_n(\boldsymbol{\varepsilon}_n) := \left\{\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n)\right\}$ and $\mathsf{U}_{n+1}(\boldsymbol{\varepsilon}_{n+1}) := \left\{\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n),\mathsf{F}_2(\varepsilon_{n+1})\right\}$, where $\boldsymbol{\varepsilon}_n := \left(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\right)$, $\boldsymbol{\varepsilon}_{n+1} := \left(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,\varepsilon_{n+1}\right)$. Let us assume that we have the Golay matrix $\mathbf{G}_{2^n}^{[n]}\left[\boldsymbol{\varepsilon}_n\right] = \mathbf{G}_{2^n}^{[n]}\left[\mathsf{U}_n(\boldsymbol{\varepsilon}_n)\right] = \mathbf{G}_{2^n}^{[n]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n)\right]$ (depending on n previous transforms $\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n)$. We need to construct the next Golay matrix $\mathbf{G}_{2^n}^{[n+1]}\left[\boldsymbol{\varepsilon}_{n+1}\right] = \mathbf{G}_{2^{n+1}}^{[n+1]}\left[\mathsf{U}_n(\boldsymbol{\varepsilon}_{n+1})\right] =$ $= \mathbf{G}_{2^n}^{[n]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n),\mathsf{F}_2(\varepsilon_{n+1})\right]$ using only $\mathbf{G}_{2^n}^{[n]}\left[\mathsf{U}_n(\boldsymbol{\varepsilon}_n)\right]$ and $\mathsf{F}_2(\varepsilon_{n+1})$. We are going to use for Golay matrix $\mathbf{G}_{2^n}^{[n]}\left[\boldsymbol{\varepsilon}_n\right] = \mathbf{G}_{2^n}^{[n]}\left[\mathsf{U}_n(\boldsymbol{\varepsilon}_n)\right] = \; = \mathbf{G}_{2^n}^{[n]}\left[\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),\ldots,\mathsf{F}_2(\varepsilon_n)\right]$ the same structure as in (2)

$$\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\varepsilon}_n) = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) \end{bmatrix}. \tag{8}$$

For constructing $\mathbf{G}_{2^n}^{[n+1]}\left[\boldsymbol{\varepsilon}_{n+1}\right]$ from $\mathbf{G}_{2^n}^{[n]}\left[\boldsymbol{\varepsilon}_n\right]$ we take each complementary pair in the form of $\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) \end{bmatrix}$ from (8). The Golay $\left(2^{n+1} \times 2^{n+1}\right)$-matrix $\mathbf{G}_{2^n}^{[n+1]}\left[\boldsymbol{\varepsilon}_{n+1}\right]$ is constructed according to the following iteration rule

$$\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \end{bmatrix}^{\mathsf{Z}}_{]} \begin{bmatrix} {}^{(0)}\mathsf{F}(\varepsilon_{n+1}) \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \end{bmatrix} \\ \hline {}^{(1)}\mathsf{F}(\varepsilon_{n+1}) \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \varepsilon_{n+1}\mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\varepsilon_{n+1}\mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n\right) \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) + \varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n \,|\, \boldsymbol{\varepsilon}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) - \varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n \,|\, \boldsymbol{\varepsilon}_n) \\ \hline \varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n \,|\, \boldsymbol{\varepsilon}_n) \\ -\varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \,|\, \boldsymbol{\varepsilon}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n \,|\, \boldsymbol{\varepsilon}_n) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) \end{bmatrix},$$

where

$$\begin{aligned} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) &= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \boldsymbol{\varepsilon}_{n+1} \cdot \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) &= \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \boldsymbol{\varepsilon}_{n+1} \cdot \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) &= \boldsymbol{\varepsilon}_{n+1} \cdot \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n), \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1} \,|\, \boldsymbol{\varepsilon}_{n+1}) &= -\boldsymbol{\varepsilon}_{n+1} \cdot \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n). \end{aligned}$$

are complementary sequences of twice length, belonging to $\mathbf{G}_{2^n}^{[n+1]}\left[\boldsymbol{\varepsilon}_{n+1}\right]$. Hence,

$$\mathbf{G}_{2^{n+1}}^{[n+1]}(\boldsymbol{\varepsilon}_{n+1})=\mathbf{G}_{2^{n+1}}^{[n+1]}(\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})=\mathop{\boxplus}_{\boldsymbol{\alpha}_{\mathbf{n+1}}=\mathbf{0}}^{\mathbf{2^{n+1}-1}}\left[\mathrm{com}_{\boldsymbol{\alpha}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\right]=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n-1}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\end{bmatrix}=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1}\left[\begin{array}{c}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})\end{array}\right]=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{\mathbf{2^{n-1}-1}}\left[\begin{array}{c}\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & \varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1}\\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & -\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\end{bmatrix}\\ \hdashline \begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & \varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0}\\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & -\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\end{bmatrix}\end{array}\right]=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{\mathbf{2^{n-1}-1}}\left(\mathop{\boxplus}_{\alpha_n=0}^{1}\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\oplus_2\alpha_n)} & \varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\oplus_2\alpha_n)}\\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\oplus_2\alpha_n)} & -\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\oplus_2\alpha_n)}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)\end{bmatrix}\right)=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{\mathbf{2^{n-1}-1}}\left(\mathop{\boxplus}_{\alpha_n=0}^{1}\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\oplus_2\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)+\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\oplus_2\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0\oplus_2\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)-\varepsilon_{n+1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1\oplus_2\alpha_n)}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}\right)=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1}\left[\begin{array}{c}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)+\varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n+2^n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)-\varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n+2^n)\\ \hline \varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)+\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n+2^n)\\ -\varepsilon_{n+1}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)+\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n+2^n)\end{array}\right], \tag{9}$$

or

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_{n+1})=\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}\mid\boldsymbol{\varepsilon}_{n+1})=$$

$$=\varepsilon_{n+1}^{0}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(\alpha_n\oplus_2 0)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)+(-1)^{\alpha_{n+1}}\varepsilon_{n+1}^{1}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(\alpha_n\oplus_2 1)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)=$$

$$=\sum_{\beta_n=0}^{1}(-1)^{\alpha_{n+1}\beta_n}\varepsilon_{n+1}^{\beta_n}\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(\alpha_n\oplus_2\beta_n)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)=$$

$$=\sum_{\beta_n=0}^{1}(-1)^{\alpha_{n+1}\beta_n}\varepsilon_{n+1}^{\beta_n}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}^{[n]}\left(\mathbf{t}_n+2^n\cdot(\alpha_n\oplus_2\beta_n)\mid\boldsymbol{\varepsilon}_n\right).$$

Since, $\mathbf{t}_{n+1}=(\mathbf{t}_n,t_{n+1})$ then believing $t_{n+1}=\alpha_n\oplus_2\beta_n$, we obtain

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n,t_{n+1}\mid\boldsymbol{\varepsilon}_{n+1})=\sum_{t_{n+1}=0}^{1}(-1)^{\left(\alpha_n\oplus_2 t_{n+1}\right)\alpha_{n+1}}\varepsilon_{n+1}^{\alpha_n\oplus_2 t_{n+1}}\cdot\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot t_{n+1}}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n)=$$

$$=\sum_{t_{n+1}=0}^{1}(-1)^{\left(\alpha_n\oplus_2 t_{n+1}\right)\alpha_{n+1}}\varepsilon_{n+1}^{\alpha_n\oplus_2 t_{n+1}}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}^{[n]}\left(\mathbf{t}_n+2^n\cdot t_{n+1}\mid\boldsymbol{\varepsilon}_n\right)=$$

$$=(-1)^{\alpha_n\alpha_{n+1}}\sum_{t_{n+1}=0}^{1}(-1)^{t_{n+1}\alpha_{n+1}}\varepsilon_{n+1}^{\alpha_n\oplus_2 t_{n+1}}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}^{[n]}\left(\mathbf{t}_n+2^n\cdot t_{n+1}\mid\boldsymbol{\varepsilon}_n\right).$$

Therefore,

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n,t_{n+1}\mid\boldsymbol{\varepsilon}_n,\varepsilon_{n+1})=(-1)^{\left(\alpha_n\oplus_2 t_{n+1}\right)\alpha_{n+1}}\varepsilon_{n+1}^{\alpha_n\oplus_2 t_{n+1}}\cdot\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}^{[n]}(\mathbf{t}_n\mid\boldsymbol{\varepsilon}_n). \tag{10}$$

The Golay$\left(2^{n+1}\times 2^{n+1}\right)$-matrix $\mathbf{G}_{2^n}^{[n+1]}\left[\boldsymbol{\varepsilon}_{n+1}\right]$ is multiparameter matrix, depending on $n+1$ parameters $\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_n,\varepsilon_{n+1}\right)$. It is easy to proof, that sequences (10) are complementary and unitary sequences.

Example 2. Let us construct $\mathbf{G}_{2^1}^{[1]}[\boldsymbol{\varepsilon}_1]$, $\mathbf{G}_{2^2}^{[2]}[\boldsymbol{\varepsilon}_2]$ and $\mathbf{G}_{2^3}^{[3]}[\boldsymbol{\varepsilon}_3]$:

$$\mathbf{G}_{2^1}^{[1]}(\varepsilon_1) \equiv \mathsf{F}_2(\varepsilon_1) = \left[\mathrm{com}_{(\alpha_1)}^{[1]}(\mathbf{t}_1 \mid \varepsilon_1\right] = \begin{bmatrix} \mathrm{com}_{(0)}^{[1]}(\mathbf{t}_1 \mid \varepsilon_1) \\ \mathrm{com}_{(1)}^{[1]}(\mathbf{t}_1 \mid \varepsilon_1) \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon_1 \\ 1 & -\varepsilon_1 \end{bmatrix} = \left[(-1)^{\alpha_1 t_1}\varepsilon^{t_1}\right] =$$

$$= \left[(-1)^{\alpha_1 t_1}\right] \cdot \mathrm{diag}\left\{\varepsilon^{t_1}\right\} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & \varepsilon_1 \end{bmatrix} = \mathbf{G}_{2^1}^{[1]} \cdot \mathrm{diag}\left\{\varepsilon_1^{t_1}\right\},$$

$$\mathbf{G}_{2^2}^{[2]}(\varepsilon_1,\varepsilon_2) = \begin{bmatrix} \mathrm{com}_{(0,0)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2) \\ \mathrm{com}_{(0,1)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2) \\ \hline \mathrm{com}_{(1,0)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2) \\ \mathrm{com}_{(1,1)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & \varepsilon_1 & \varepsilon_2 & -\varepsilon_2\varepsilon_1 \\ 1 & \varepsilon_1 & -\varepsilon_2 & \varepsilon_2\varepsilon_1 \\ \hline \varepsilon_2 & -\varepsilon_2\varepsilon_1 & 1 & \varepsilon_1 \\ -\varepsilon_2 & \varepsilon_2\varepsilon_1 & 1 & \varepsilon_1 \end{array}\right] = \left[\mathrm{com}_{(\alpha_1,\alpha_2)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2)\right] =$$

$$= \left[\mathrm{com}_{(\alpha_1 \underset{2}{\oplus} t_2,\alpha_2)}^{[1]}(t_1 \mid \varepsilon_1)(-1)^{(\alpha_1 \underset{2}{\oplus} t_2)\alpha_2}\varepsilon_2^{(\alpha_1 \underset{2}{\oplus} t_2)}\right] = \left[(-1)^{(\alpha_1 \underset{2}{\oplus} t_2)t_1}\varepsilon^{t_1}(-1)^{(\alpha_1 \underset{2}{\oplus} t_2)\alpha_2}\varepsilon_2^{(\alpha_1 \underset{2}{\oplus} t_2)}\right] =$$

$$= \mathrm{diag}\left\{\varepsilon_2^{\alpha_1}\right\}\left[(-1)^{(\alpha_1 \underset{2}{\oplus} t_2)t_1}(-1)^{(\alpha_1 \underset{2}{\oplus} t_2)\alpha_2}\varepsilon_2^{(\alpha_1 \underset{2}{\oplus} t_2)}\right]\mathrm{diag}\left\{\varepsilon_1^{t_1}\varepsilon_2^{t_2}\right\} = \mathrm{diag}\left\{\varepsilon_2^{\alpha_1}\right\} \cdot \mathbf{G}_{2^2}^{[2]} \cdot \mathrm{diag}\left\{\varepsilon_1^{t_1}\varepsilon_2^{t_2}\right\} =$$

$$= \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & \varepsilon_2 & \\ & & & \varepsilon_2 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right]\left[\begin{array}{cc|cc} 1 & & & \\ & \varepsilon_1 & & \\ \hline & & \varepsilon_2 & \\ & & & \varepsilon_1\varepsilon_2 \end{array}\right],$$

$$\mathbf{G}_{2^3}^{[3]}(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \left[\mathrm{com}_{(\alpha_1,\alpha_2,\alpha_2)}^{[3]}(\mathbf{t}_3 \mid \varepsilon_1,\varepsilon_2,\varepsilon_3)\right] = \left[\mathrm{com}_{(\alpha_1,\alpha_2 \underset{2}{\oplus} t_3)}^{[2]}(\mathbf{t}_2 \mid \varepsilon_1,\varepsilon_2)(-1)^{(\alpha_2 \underset{2}{\oplus} t_3)\alpha_3}\varepsilon_2^{(\alpha_2 \underset{2}{\oplus} t_3)}\right] =$$

$$= \mathrm{diag}\left\{\varepsilon_2^{\alpha_1}\varepsilon_3^{\alpha_2}\right\} \cdot \mathbf{G}_{2^3}^{[3]} \cdot \mathrm{diag}\left\{\varepsilon_1^{t_1}\varepsilon_2^{t_2}\varepsilon_3^{t_3}\right\}.$$

The resulting matrix still has the orthogonal rows and every pair is complementary in the Golay-Rudin-Shapiro sense. From (10) we see that

$$\mathbf{G}_{2^n}^{[n]}(\varepsilon_1,\varepsilon_2,...,\varepsilon_n) = \left[\mathrm{com}_{(\alpha_1,\alpha_2,..,\alpha_n)}^{[n]}(\mathbf{t}_n \mid \varepsilon_1,\varepsilon_2,...,\varepsilon_n)\right] = \mathrm{diag}\left\{\varepsilon_2^{\alpha_1}\varepsilon_3^{\alpha_2}\cdot ...\cdot\varepsilon_n^{\alpha_{n-1}}\right\} \cdot \mathbf{G}_{2^n}^{[n]} \cdot \mathrm{diag}\left\{\varepsilon_1^{t_1}\varepsilon_2^{t_2}\cdot ...\cdot\varepsilon_n^{t_n}\right\},$$

$$\mathrm{com}_{(\alpha_1,\alpha_2,..,\alpha_n)}^{[n]}(\mathbf{t}_n \mid \varepsilon_1,\varepsilon_2,...,\varepsilon_n) = \prod_{i=1}^{n-1}\varepsilon_{i+1}^{\alpha_i} \cdot \left[\mathrm{com}_{(\alpha_1,\alpha_2,..,\alpha_n)}^{[n]}(\mathbf{t}_n)\right] \cdot \prod_{i=1}^{n}\varepsilon_i^{t_i}. \tag{11}$$

If $\varepsilon_1 = e^{j\theta_1}$, $\varepsilon_2 = e^{j\theta_2}$,..., $\varepsilon_n = e^{j\theta_n} \in \mathbf{C}$ are complex numbers, then $\mathbf{G}_{2^n}$ is the complex-valued ($\mathbf{C}$-valued) Fourier-Golay-Rudin-Shapiro transform (FGRST).

If $\varepsilon_1, \varepsilon_2,..., \varepsilon_n \in \mathbf{GF}(p)$, tnen $\mathbf{G}_{2^n}^{[n]}(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$ is the number theoretical Galois-Golay-Rudin-Shapiro transform (GGRS-NTT), if $\varepsilon_1, \varepsilon_2,..., \varepsilon_{n+1} \in Clif$, where $Clif$ is the Clifford algebra, then $\mathbf{G}_{2^n}^{[n]}(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$ is the Clifford-Golay-Rudin-Shapiro transform, if $\varepsilon_1, \varepsilon_2,..., \varepsilon_{n+1} \in Ham$, where Ham is the quaternion Hamilnon algebra, then $\mathbf{G}_{2^n}^{[n]}(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$ is the Hamilton-Golay-Rudin-Shapiro transform and so on.

5. Conclusion

In this paper, we have shown a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-,or Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n,\mathsf{F}_2,\mathbf{C}\right)$, but with $\left(\mathbf{Z}_2^n,\left\{\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),...,\mathsf{F}_2(\varepsilon_n)\right\},\mathsf{A}lg\right)$, where $\left\{\mathsf{F}_2(\varepsilon_1),\mathsf{F}_2(\varepsilon_2),...,\mathsf{F}_2(\varepsilon_n)\right\}$ a set of arbitrary unitary (2×2)-transforms of type $\mathsf{F}_2(\varepsilon) = \begin{bmatrix} 1 & \varepsilon \\ 1 & -\varepsilon \end{bmatrix}$, (where $\varepsilon := e^{i\theta} \in \mathsf{A}lg$, $|\varepsilon| = 1$) instead of

$\mathsf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathsf{A}lg$ is an algebras (for example, Clifford algebras or finite rings $\mathbf{Z}_N$, or finite Galois fields $\mathbf{GF}(q)$) instead of the complex field $\mathbf{C}$.

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